Nonsmooth composite minimization:

an exponentially convergent primal-dual algorithm

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joint work with

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Nonsmooth composite minimization

$$\underset{x}{\text{minimize}} \qquad \underset{\textbf{performance}}{\underset{\Big\downarrow}{f(x)}} \qquad + \qquad \underset{\textbf{structure}}{\underset{\Big\downarrow}{g(Tx)}}$$

Nonsmooth composite minimization

$$\underset{x}{\text{minimize}} \qquad \underset{\substack{\big\downarrow \\ \textbf{performance}}}{f(x)} \qquad + \qquad \underset{\substack{\big\downarrow \\ \textbf{structure}}}{g(Tx)}$$

- T – select certain coordinates to impose structure
- f – strongly convex; Lipschitz cts gradient
- g – non-differentiable; convex

  $e.g., I_{\mathcal{C}}(\cdot), \|\cdot\|_1, \|\cdot\|_*$, easy to evaluate proximal operator

Proximal gradient descent

$$\underset{x}{\text{minimize}} \qquad \underset{\substack{\downarrow \\ \text{smooth}}}{f(x)} \qquad + \qquad \underset{\substack{\downarrow \\ \text{nonsmooth}}}{g(x)}$$

Beck & Teboulle, SIAM J. Imag. Sci. ’09
Parikh & Boyd, FnT in Optimization ’14

Proximal gradient descent

$$\underset{x}{\text{minimize}} \qquad \underset{\text{smooth}}{\underset{\downarrow}{f(x)}} \qquad + \qquad \underset{\text{nonsmooth}}{\underset{\downarrow}{g(x)}}$$

- $\mathbf{prox}_{\mu g}(v) \;:=\; \underset{z}{\operatorname{argmin}}\; g(z) + \frac{1}{2\mu}\|z - v\|^2$

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- **Gradient descent plus proximal operator**

$$x^{k+1} \; = \; \mathbf{prox}_{\alpha_k g}(x^k - \alpha_k \nabla f(x^k))$$

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  - $\mathbf{prox}_{\mu g}(v) \;:=\; \underset{z}{\text{argmin}}\; g(z) + \frac{1}{2\mu}\|z - v\|^2$
  - explicit formula for $\mathbf{prox}_g \longrightarrow$ efficient implementation
  - does not apply to $g(Tx)$

Beck & Teboulle, SIAM J. Imag. Sci. ’09

Parikh & Boyd, FnT in Optimization ’14

$$\begin{array}{ll} \underset{x,z}{\text{minimize}} & f(x) + g(z) \\ \text{subject to} & T\,x - z = 0 \end{array}$$

Boyd, Parikh, Chu, Peleato, Eckstein, FnT in Optimization ’11

Augmented Lagrangian method

$$\begin{aligned} \underset{x,z}{\text{minimize}} \quad & f(x) + g(z) \\ \text{subject to} \quad & T\,x - z = 0 \end{aligned}$$

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- **Augmented Lagrangian**

$$\mathcal{L}_\mu(x, z; y) = f(x) + g(z) + y^T(Tx - z) + \frac{1}{2\mu}\|Tx - z\|^2$$

Boyd, Parikh, Chu, Peleato, Eckstein, FnT in Optimization ’11

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- **ADMM**

$$\begin{aligned} x^{k+1} &= \underset{x}{\text{argmin}}\ \mathcal{L}_\mu(x, z^k; y^k) \\ z^{k+1} &= \underset{z}{\text{argmin}}\ \mathcal{L}_\mu(x^{k+1}, z; y^k) \qquad \mathbf{prox}_{\mu g}(\cdot) \\ y^{k+1} &= y^k + \frac{1}{\mu}(Tx^{k+1} - z^{k+1}) \end{aligned}$$

Boyd, Parikh, Chu, Peleato, Eckstein, FnT in Optimization ’11

Proximal augmented Lagrangian

$$\mathcal{L}_\mu(x, z; y) \;=\; f(x) \;+\; g(z) \;+\; \frac{1}{2\mu}\|z - (Tx + \mu y)\|^2 \;-\; \frac{\mu}{2}\|y\|^2$$

Dhingra, Khong, Jovanović, IEEE TAC ’18

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- **Minimizer of $\mathcal{L}_\mu(x, z; y)$ over z**

$$z_\mu^\star(x,\, y) \;=\; \mathbf{prox}_{\mu g}(Tx + \mu y)$$

Dhingra, Khong, Jovanović, IEEE TAC '18

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- **Evaluate $\mathcal{L}_\mu(x,z;y)$ at $z_\mu^\star$**

$$\mathcal{L}_\mu(x;y) \quad := \quad \mathcal{L}_\mu(x,z;y)\Big|_{z\,=\,z_\mu^\star}$$

Dhingra, Khong, Jovanović, IEEE TAC ’18

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$$\begin{aligned} \mathcal{L}_\mu(x;y) \quad &:= \quad \mathcal{L}_\mu(x,z;y)\Big|_{z\,=\,z_\mu^\star} \\ &= \quad f(x) \;+\; M_{\mu g}(Tx + \mu y) \;-\; \tfrac{\mu}{2}\|y\|^2 \end{aligned}$$

continuously differentiable in x and y

Dhingra, Khong, Jovanović, IEEE TAC '18

Primal-dual gradient flow dynamics

- **Primal-descent dual-ascent**

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -\nabla_x \mathcal{L}_\mu(x;y) \\ \nabla_y \mathcal{L}_\mu(x;y) \end{bmatrix}$$

$$= \begin{bmatrix} -(\nabla f(x) + T^T \nabla M_{\mu g}(Tx + \mu y)) \\ \mu(\nabla M_{\mu g}(Tx + \mu y) - y) \end{bmatrix}$$

$$\mu \nabla M_{\mu g}(v) \; = \; v \; - \; \mathbf{prox}_{\mu g}(v)$$

- Lipschitz cts RHS
- $\dot{x} = 0, \dot{y} = 0$ – optimality condition

$$\left.\begin{array}{l} \bullet \; f \text{ – strongly convex; Lipschitz cts gradient} \\ \bullet \; T \text{ – row full rank} \end{array}\right\} \rightarrow \text{exponentially stable}$$

Dhingra, Khong, Jovanović, IEEE TAC '18

Implementation issues

- **Key issue**
  - how do we implement it? discretization
- **Simple discretization**

$$\begin{bmatrix} x^{k+1} \\ y^{k+1} \end{bmatrix} = \begin{bmatrix} x^k - \alpha \nabla_x \mathcal{L}_\mu(x^k; y^k) \\ y^k + \alpha \nabla_y \mathcal{L}_\mu(x^k; y^k) \end{bmatrix}$$

- **Key challenge**
  - CT convergence rate analysis ⟶ DT algorithm

Wibisono, Wilson, Jordan, PNAS '16
Wilson, Recht, Jordan, arXiv:1611.02635
Hu, Lessard, ICML '17

Proposed primal-dual algorithm

- **Forward Euler discretization**

$$\begin{bmatrix} x^{k+1} \\ y^{k+1} \end{bmatrix} = \begin{bmatrix} x^k - \alpha \nabla_x \mathcal{L}_\mu(x^k; y^k) \\ y^k + \alpha \nabla_y \mathcal{L}_\mu(x^k; y^k) \end{bmatrix}$$

$$= \begin{bmatrix} x^k - \alpha(\nabla f(x^k) + T^T \nabla M_{\mu g}(Tx^k + \mu y^k)) \\ y^k + \alpha\mu(\nabla M_{\mu g}(Tx^k + \mu y^k) - y^k) \end{bmatrix}$$

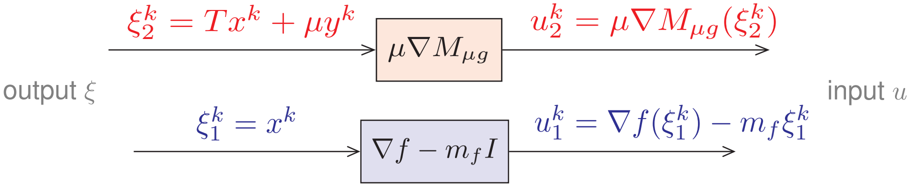
- **Contributions**
  - an automated tool ⟶ exp. converg. of PD algorithm
  - LMI condition ⟶ rate certificate
  - a range of step size values ⟶ exp. converg.

Lessard, Recht, Packard, SIAM J. Optim. ’16
Hu, Seiler, Rantzer, PMLR ’17
Fazlyab, Ribeiro, Morari, Preciado, SIAM J. Optim. ’18

Control-theoretic viewpoint

- **Feedback connection**

$$\begin{aligned}\begin{bmatrix} x^{k+1} \\ y^{k+1} \end{bmatrix} \;&=\; \begin{bmatrix} (1-\alpha m_f)x^k \\ (1-\alpha\mu)y^k \end{bmatrix} \\ &\quad -\alpha \begin{bmatrix} \nabla f(x^k) - m_f x^k \\ 0 \end{bmatrix} \\ &\quad -\alpha \begin{bmatrix} T^T \nabla M_{\mu g}(Tx^k + \mu y^k) \\ -\mu \nabla M_{\mu g}(Tx^k + \mu y^k) \end{bmatrix}\end{aligned}$$



$\xi_2^k = Tx^k + \mu y^k$

$\mu \nabla M_{\mu g}$

$u_2^k = \mu \nabla M_{\mu g}(\xi_2^k)$

output ξ

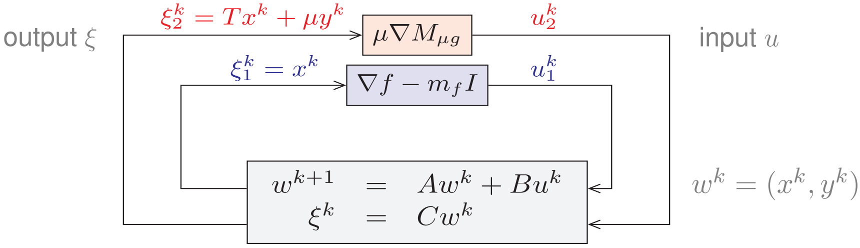
input u

$\xi_1^k = x^k$

$\nabla f - m_f I$

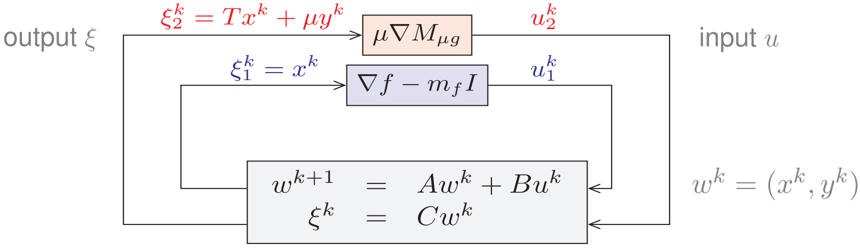
$u_1^k = \nabla f(\xi_1^k) - m_f \xi_1^k$

Control-theoretic viewpoint



$u_1^k \;=\; \nabla f(\xi_1^k) - m_f \xi_1^k \qquad\qquad u_2^k \;=\; \mu \nabla M_{\mu g}(\xi_2^k)$

Control-theoretic viewpoint



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$\mu \nabla M_{\mu g}$

u_2^k

input u

$\xi_1^k = x^k$

$\nabla f - m_f I$

u_1^k

$$\begin{aligned} w^{k+1} &= Aw^k + Bu^k \\ \xi^k &= Cw^k \end{aligned}$$

$w^k = (x^k, y^k)$

$$u_1^k = \nabla f(\xi_1^k) - m_f \xi_1^k \qquad u_2^k = \mu \nabla M_{\mu g}(\xi_2^k)$$

- **LTI system:** (A, B, C)

$$A = \begin{bmatrix} (1 - \alpha m_f)I & 0 \\ 0 & (1 - \alpha\mu)I \end{bmatrix}$$

$$B = \begin{bmatrix} -\alpha I & -\frac{\alpha}{\mu} T^T \\ 0 & \alpha I \end{bmatrix}, C = \begin{bmatrix} I & 0 \\ T & \mu I \end{bmatrix}$$

Exponential convergence

- **Exponential/linear convergence with rate $r \in (0,1)$**

$$\begin{aligned} \begin{bmatrix} A^T P A - r^2 P & A^T P B \\ B^T P A & B^T P B \end{bmatrix} + \begin{bmatrix} C^T & 0 \\ 0 & I \end{bmatrix} \Pi \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} &\prec 0 \\ P &\succ 0 \end{aligned}$$

Lessard, Recht, Packard, SIAM J. Optim. ’16

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$w^k = (x^k, y^k)$

$$\Downarrow$$

$$\|w^k - \bar{w}\| \;\leq\; \sqrt{\mathsf{cond}(P)}\, r^k \, \|w^0 - \bar{w}\|$$

Lessard, Recht, Packard, SIAM J. Optim. '16

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$$P \succ 0$$

$$\Downarrow$$

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$$\|w^k - \bar{w}\| \leq \sqrt{\text{cond}(P)}\, r^k \, \|w^0 - \bar{w}\|$$

- (A, B, C) – algorithm parameters (α, μ, T, m_f)
- Π – problem parameters (L_f, m_f)
- $(\alpha, r) \longrightarrow$ decision variable P
- $r \in (0,1) \longrightarrow$ decision variables (α, P)

Lessard, Recht, Packard, SIAM J. Optim. '16

Exponential convergence

- **Exponential/linear convergence with rate** $r \in (0,1)$

$$\begin{aligned} \begin{bmatrix} G(r\mathrm{e}^{\mathrm{j}\theta}) \\ I \end{bmatrix}^* \Pi \begin{bmatrix} G(r\mathrm{e}^{\mathrm{j}\theta}) \\ I \end{bmatrix} &\prec 0, \ \forall\, \theta \in [0, 2\pi) \\ G(r\mathrm{e}^{\mathrm{j}\theta}) &\in \mathcal{RH}_\infty \end{aligned}$$

Rantzer, Syst. Control Lett. '96

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Rantzer, Syst. Control Lett. '96

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$$G(r\mathrm{e}^{\mathrm{j}\theta}) \in \mathcal{RH}_\infty$$

$$w^k = (x^k, y^k)$$

$$\Downarrow$$

$$\|w^k - \bar{w}\| \leq c\, r^k \, \|w^0 - \bar{w}\|$$

- $G(r\mathrm{e}^{\mathrm{j}\theta}) = C(r\mathrm{e}^{\mathrm{j}\theta} I - A)^{-1} B$ – transfer function
- Π – problem parameters (L_f, m_f)
- no need to find P

Rantzer, Syst. Control Lett. ’96

Sketch of the derivation

- **Choose $\mu = L_f - m_f$ and rewrite**

$$\begin{bmatrix} a(\zeta)I & b(\zeta)T^T \\ b(\zeta)T & c(\zeta)I + d(\zeta)TT^T \end{bmatrix} \succ 0, \forall \zeta \in [-1, 1]$$

  - functions $a(\zeta), b(\zeta), c(\zeta)$, and $d(\zeta)$ are parameterized by (α, r)

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- **Take Schur complement and impose stability**
  - $a(\zeta) > 0, \; c(\zeta) + \left(d(\zeta) - \frac{b^2(\zeta)}{a(\zeta)} \right) TT^T \succ 0, \; \forall \, \zeta \in [-1, 1]$

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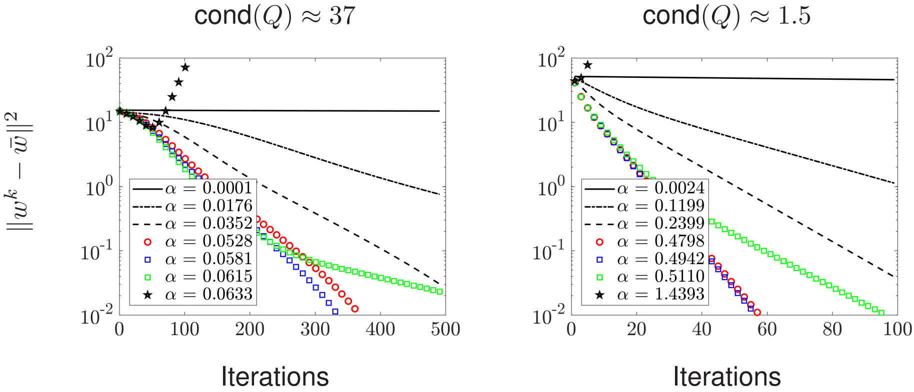
$$\left.\begin{array}{l} \cdot \; f \text{ -- } m_f\text{-strongly convex} \\ \cdot \; \nabla f \text{ -- } L_f\text{-Lipschitz cts} \\ \cdot \; T \text{ -- row full rank} \end{array}\right\} \rightarrow \begin{array}{l} \text{Exponentially convergent if} \\ \quad \cdot \; \mu \geq L_f - m_f \\ \quad \cdot \; 0 < \alpha < \bar{\alpha}(\mu, m_f, L_f, \lambda_m(TT^T)) \end{array}$$

Quadratic programming

$$\begin{array}{ll} \underset{x,\,z}{\text{minimize}} & \frac{1}{2}\,x^T Q x \,+\, q^T x \,+\, g(z) \\ \text{subject to} & Tx \,-\, z \,=\, 0. \end{array}$$


cond$(Q) \approx 37$

cond$(Q) \approx 1.5$

Nishihara, Lessard, Recht, Packard, Jordan, ICML ’15

Summary

- **Results**
  - an iterative primal-dual algorithm
  - an automated tool
  - a region of step size with exp. converg.

- **Ongoing works**
  - reduce the conservative
  - other discretization methods

THANK YOU!